CS 312: Algorithm Analysis

Lecture #37: Linear Programming: Simplex Method

Announcements

- Need Thought
- Help Session on Proj. #7 on Wednesday
- Any questions?

Objectives

Lecture #36: Linear Programming: Slack Form

- Review LP problem formulation
- Convert to Standard Form
- Convert to Slack Form

Lecture #37: Linear Programming: Simplex Method

- Quickly Review Slack Form
- Introduce Pivot Algorithm
- Walk through Example
- Understand Simplex Method

EXAMPLE IN STANDARD FORM

$$\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Convert to Slack Form

$$
\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & = & z \\
\text{subject to} & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$C_1: \quad x_4 = 30 - (x_1 + x_2 + 3x_3)$

$C_2: \quad x_5 = 24 - (2x_1 + 2x_2 + 5x_3)$

$C_3: \quad x_6 = 36 - (4x_1 + x_2 + 2x_3)$

$x_4, x_5, x_6 \geq 0$

Basic Solution

$$
\begin{aligned}
z &= & & 3x_1 &+& x_2 &+& 2x_3 \\
x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
x_6 &= 36 &-& 4x_1 &-& x_2 &-& 2x_3
\end{aligned}
$$

$$\bar{\underline{x}} = (\underbrace{\bar{x}_1, \bar{x}_2, \bar{x}_3}_{\text{NON-BASIC}}, \underbrace{\bar{x}_4, \bar{x}_5, \bar{x}_6}_{\text{BASIC}})$$

$$N = \{1, 2, 3\} \qquad B = \{4, 5, 6\}$$

$$= (0, 0, 0, 30, 24, 36)$$

"BASIC SOL'N"

Identify a Non-Basic Variable

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

$\bar{x} = (0, 0, 0, 30, 24, 36)$

CHOOSE x_1 BECAUSE ITS COEFF. IN THE OBJECTIVE FUNCTION IS HIGHEST.

$x_1 = 9$ IS HIGHEST VALUE I CAN CHOOSE WITHOUT VIOLATING A CONSTRAINT.

$\bar{x} = (9, 0, 0, 21, 6, 0)$

Select a Basic Variable

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

$\bar{x} = (9, 0, 0, 21, 6, 0)$

C_3 WAS THE TIGHTEST CONSTRAINT.

CHOOSE $\underline{x_6}$

Solve for the Non-Basic Var.

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

x_e = "ENTERING VAR."

$= x_6$

x_ℓ = "LEAVING VAR."

$= x_1$

"PIVOT"

$$4x_1 = 36 - x_2 - 2x_3 - x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

Rewrite the Other Equations

$$
\begin{aligned}
x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
4x_1 &= 36 - x_2 - 2x_3 - x_6 \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}
\end{aligned}
$$

$$
\begin{aligned}
z &= && 3x_1 &+& x_2 &+& 2x_3 \\
C_1:\ x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
C_2:\ x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
\cancel{x_6} &\cancel{= 36} &\cancel{-}& \cancel{4x_1} &\cancel{-}& \cancel{x_2} &\cancel{-}& \cancel{2x_3}
\end{aligned}
$$

$$
\begin{aligned}
z &= 3\left(9 - \tfrac{1}{4}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{4}x_6\right) + x_2 + 2x_3 \\
&= 27 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 - \tfrac{3}{4}x_6
\end{aligned}
$$

ALSO, SUBSTITUTE EXPR. FOR x_1 INTO CONSTRAINTS 1 & 2.

- UPDATE EXPR. FOR x_4 & x_5

After the Pivot: Objective Values

$$
\left.\begin{array}{rclcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}\right\}
$$

NEW LP PROBLEM!

$\bar{x} = (9, 0, 0, 21, 6, 0)$

NOTICE Z: 27 NO CHANGE!

Repeat

$$
\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

$\bar{x} = (9, 0, 0, 21, 6, 0)$

1. PICK A NON-BASIC VARIABLE

x_3 — BECAUSE IT HAS LGEST COEFF.

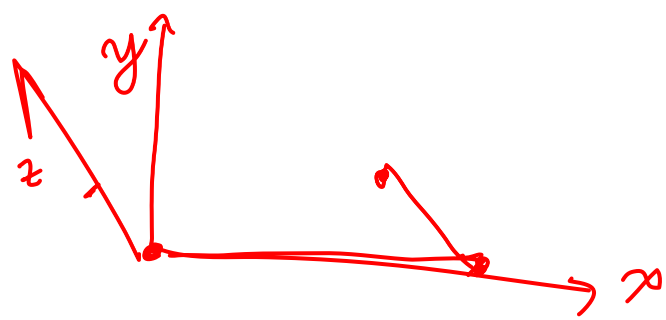
$x_3 = 1.5$

$\underline{x} = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$

2. TIGHTEST CONSTRAINT : 3RD CONSTRAINT FOR x_5

$x_e = x_5$, $x_\ell = x_3$

3. PIVOT



Repeat Again (Third time)

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

$\underline{x} = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$

1. PICK A NON-BASIC: x_2

   - RAISE IT TO $x_2 = 4$

   $\underline{x} = (8, 4, 0, 18, 0, 0)$

2. BASIC - FORCED TO CHOOSE x_3

$x_e = x_3$

$x_\ell = x_2$

PIVOT.

Objective & Final Solution

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

$= 28$

$\underline{x} = (8, 4, 0, 18, 0, 0)$

CAN'T PICK ANOTHER NON-BASIC VAR. TO GROW OBJ. FUNCTION

DONE.

RETURN SOLUTION

Pivot Algorithm

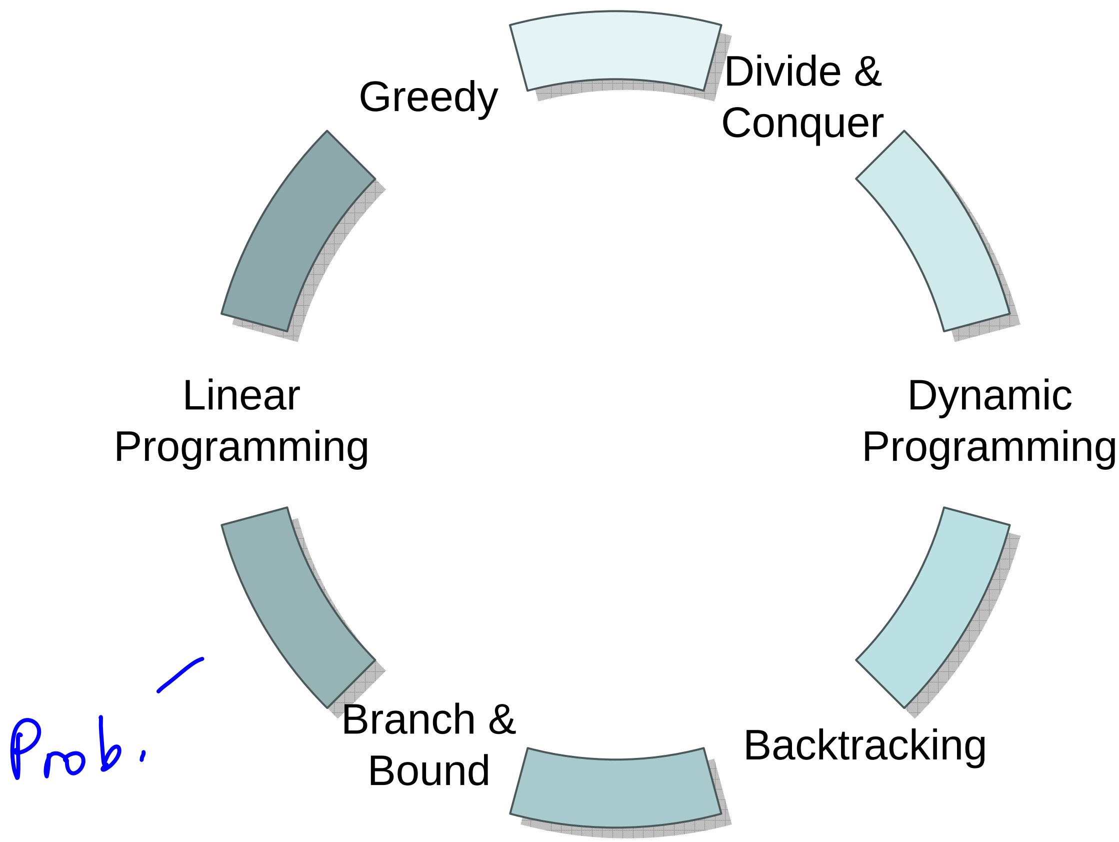
INDEX OF LEAVING VAR.
INDEX OF ENTERING VAR.

Pivot (N, B, A, b, c, v, l, e)
// Compute coefficients for equation for the new basic variable x_e
// The variables with hats, like $\hat{b}_e$, will be the return values
$\hat{b}_e = b_l / a_{l,e}$
for each $j \in N - \{e\}$
    do $\hat{a}_{e,j} = a_{l,j} / a_{l,e}$
$\hat{a}_{e,l} = 1 / a_{l,e}$
// Compute coefficients for the other constraints
for each $i \in B - \{l\}$
    do $\hat{b}_i = b_i - a_{i,e} \hat{b}_e$
        for each $j \in N - \{e\}$
            do $\hat{a}_{i,j} = a_{i,j} - a_{i,e} \hat{a}_{e,j}$
        $\hat{a}_{i,l} = -a_{i,e} \hat{a}_{e,l}$
// Compute the objective function
$\hat{v} = v + c_e \hat{b}_e$
for each $j \in N - \{e\}$
    do $\hat{c}_j = c_j - c_e \hat{a}_{e,j}$
$\hat{c}_l = -c_e \hat{a}_{e,l}$
// Compute new basic and nonbasic variable sets
$\hat{N} = (N - \{e\}) \cup \{l\}$
$\hat{B} = (B - \{l\}) \cup \{e\}$
return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$

Simplex

```
Simplex (A, b, c)
// convert the problem to slack form and check feasibility
// you may assume that the first basic solution is feasible
// this means you will only convert your problem to slack form
// this assumption is not true in general but makes InitializeSimplex trivial.
(N, B, A, b, c, v) = InitialzeSimplex(A, B, c)
while there exists some j ∈ N such that c_j > 0
    // e will be the entering variable
    do choose an index e ∈ N such that c_e > 0
        for each index i ∈ B
            do if a_{i,e} > 0
                then δ_i = b_i/a_{i,e}
                else δ_i = ∞
        // l will be the leaving variable
        choose l ∈ B such that δ_i is minimized
        if δ_l = ∞
            then return "unbounded"
            else (N, B, A, b, c, v) = Pivot (N, B, A, b, c, v, l, e)
// set the nonbasic variables to 0 and everything else to the optimal solution
for i = 1 to n
    do if i ∈ B
        then x̄_i = b_i
        else x̄_i = 0
return (x̄_1, x̄_2, ... x̄_n)
```

We've Come Full Circle



Assignment

- HW #28 – see notes on the blog

1. Why is linear programming with real-valued variables simpler than linear programming with integer-valued variables?
2. Only one of the 0-1 knapsack and continuous knapsack problems can be solved using the simplex method. Which is it?
3. Which problem is more computationally difficult: the 0-1 knapsack or the continuous knapsack problem?