CS 401R: Probabilistic Foundations of Machine Learning

Lecture #2: Review of Probability Theory (cont.)

Announcements

- Prayer
- Questionnaire
- Thought every Friday
- Quick Quiz
- Reading Report #1
  - Your Questions
- Check the schedule for early and regular due dates for:
  - Assignment #1
  - Assignment #1.5 (the second half)

Quick Quiz

1. How many penalty-free late days?
2. What is the value of submission by the early deadline?
3. What are the possible grades on a reading report?
4. Is cooperation on projects allowed?

Objectives

- Build a sound framework for representing uncertainty
- Understand the fundamentals of probability theory
- Prepare to use Bayesian networks / directed graphical models extensively

Independence

- Definition: Events $A \& B$ are independent in $P$ iff $P(A \cap B) = P(A) \cdot P(B)$
- i.e., $P$ satisfies $(A \perp B)$; denoted $P \models (A \perp B)$
- i.e., knowing $B$ does not affect $P(A)$

Independence

- Example: Weather ($W$) and Rain ($R$)
  - $P(W)$ = 0.3
  - $P(R)$ = 0.1
  - $P(W) \cdot P(R) = 0.03$
  - $P(W \cap R) = 0.02$
  - $P(W) \cdot P(R) = P(W \cap R)$
  - $P(W) \perp R$

Joint prob. of $A$ and $B$

- $P(A \cap B) = \frac{P(A \cdot B)}{P(B)}$
- Likewise, $P(B \cap A) = \frac{P(B \cdot A)}{P(A)}$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A \cap B)$

Multiplication Rule

- $P(A \cap B) = P(A) \cdot P(B | A)$
- $P(A \cap B) = P(B) \cdot P(A | B)$

Probability of $\bigcap_{i=1}^{n} A_i$

- $P(\bigcap_{i=1}^{n} A_i) = P(A_1 \cap A_2 \cap \cdots \cap A_n)$
- $P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdots P(A_n | A_1 \cap A_2 \cdots A_{n-1})$

Chain rule

- We will use this often.
Independence

- **Definition:** Events \( A \) & \( B \) are independent in \( P \) iff 
  \[ P(A \cap B) = P(A) \cdot P(B) \]
- i.e., \( P \) satisfies \( A \perp B \); denoted \( P \models (A \perp B) \)
- i.e., knowing \( B \) does not affect \( P(A) \)

\[
P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)
\]

\[
P(A) = \frac{2}{3} \quad P(B) = \frac{3}{4} \quad P(A \cap B) = \frac{1}{4}
\]

**Example #2**

- \( A \cap B = \emptyset \)
- \( P(A) \neq 0 \)
- \( P(B) \neq 0 \)
- \( P(A \cap B) = 0 = P(A) \cdot P(B) \)
- Therefore, \( (A \perp B) \)
- i.e., disjoint sets are not independent

**Upshot:** Independence is a statement about probability, not about sets

Back to our Example

- **Assume Uniform Den.**

\[
A = \{ \text{HHH}, \text{HTH}, \text{THH} \}
\]

\[
B = \{ \text{HHH}, \text{HTH}, \text{HTT} \}
\]

\[
P(A) = \frac{2}{3} \quad P(B) = \frac{3}{4} \quad P(A \cap B) = \frac{1}{4}
\]

Are \( A \) & \( B \) independent ? \( \mathbf{No} \)

Change the distribution

- Before:

  \[
  P(A) = \frac{1}{3} \quad P(B) = \frac{1}{3} \quad P(A \cap B) = \frac{1}{9}
  \]

- After:

  \[
  P(A) = \frac{1}{4} \quad P(B) = \frac{1}{4} \quad P(A \cap B) = \frac{1}{16}
  \]

Are \( A \) & \( B \) independent now ? \( \mathbf{Yes} \)

Conditional Independence

- **Definition:** Events \( A \) and \( B \) are conditionally independent of one another given event \( C \) in a distribution \( P \) iff 
  \[ P(A \cap B | C) = P(A | C) \cdot P(B | C) \]
- Denoted \( P \models (A \perp B | C) \)
- Equivalent to: \( P(A | C) = P(A | B \cap C) \)
- i.e., knowing \( B \) does not affect \( P(A | C) \) in the presence of knowledge of \( C \)

Before and After

- **Assume Uniform Den.**

\[
A = \{ \text{HHH}, \text{HTH}, \text{THH} \}
\]

\[
B = \{ \text{HHH}, \text{HTH}, \text{HTT} \}
\]

\[
P(A) = \frac{2}{3} \quad P(B) = \frac{3}{4} \quad P(A \cap B) = \frac{1}{4}
\]

Are \( A \) & \( B \) independent ? \( \mathbf{No} \)

- **Assume Uniform Den.**

\[
A = \{ \text{HHH}, \text{HTH}, \text{THH} \}
\]

\[
B = \{ \text{HHH}, \text{HTH}, \text{HTT} \}
\]

\[
P(A) = \frac{2}{3} \quad P(B) = \frac{3}{4} \quad P(A \cap B) = \frac{1}{4}
\]

Are \( A \) & \( B \) independent ? \( \mathbf{No} \)

i.e., overlapping sets can be independent!
Another Example

\[ p(A) = \frac{1}{4}, \quad p(A \cap B) = \frac{1}{2} \]

\[ p(B) = \frac{1}{2}, \quad p(C) = \frac{1}{2} \]

\[ \Omega : \omega_1, \omega_2, \omega_3, \omega_4 \]

Are A and B conditionally independent, given C?

Conditionally Independent, Given C?

\[ p(A) = \frac{1}{4}, \quad p(A \cap B) = \frac{1}{2} \]

\[ p(B) = \frac{1}{2}, \quad p(C) = \frac{1}{2} \]

\[ \Omega : \omega_1, \omega_2, \omega_3, \omega_4 \]

\[ p(A \cap C) = 1, \quad p(A \cap C) = 1 \]

\[ p(A \cap B \cap C) = \frac{p(A \cap B \cap C) - p(A \cap B)}{p(C)} = 1 \]

\[ p(A \cap B \cap C) = p(A \cap B \cap C) \]

Yes!

What’s Next?

- More Probability Theory
  - Bayes Theorem
  - Marginalization
  - Random Variables
  - Etc.
- Important Ideas from Information Theory
- Bayesian Networks
- Joint (Generative) Models!

Assignment

- Assignment #1
  - Submit early for the bonus!