Announcements

- Homework #18
  - Due now

- Important: LP Notes available on the schedule
  - Work through the ideas up through Simplex algorithm
  - Include the pseudo-code for Simplex (and Pivot)
  - Set you up to succeed on project #6

- Project #5
  - Today: Early Day
  - Wednesday: Due Date

Objectives

- Introduce Linear Programming (LP)
  - Another sledge-hammer!

- Understand how to formulate a problem as an LP problem
- Represent LP problems in Standard Form
- Prepare to solve the problem using the Simplex algorithm

Example

- Objective Function
  \[ \text{Maximize: } z = -5x_1 + 3x_2 + 7x_5 \]

- Constraints
  \[ \begin{align*}
  x_1 - x_5 & \geq 2 \\
  x_1 + 2x_2 + x_3 & \leq 2 \\
  5x_1 - 2x_2 + x_3 & \geq 7 \\
  x_1, x_2, x_3 & \geq 0
  \end{align*} \]

Example #1: Profit Maximization in a chocolate shop

- Two chocolate gift products:
  - Type 1: $1 / box
  - Type 2: $6 / box

- Variables:
  - \( x_1 = \text{number of boxes of type 1 produced per day} \)
  - \( x_2 = \text{number of boxes of type 2 produced per day} \)

- Maximize profit!

- Objective:
  \[ \text{Maximize } z = x_1 + 6x_2 \]
Example #1: Profit Maximization in a chocolate shop

Variables:
\[ x_1 = \text{number of boxes of type 1 produced per day} \]
\[ x_2 = \text{number of boxes of type 2 produced per day} \]

Constraints:
\[ x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

Level Sets

Which points in the feasible region should we care about?

• Start with objective function:
\[ x_1 + 6x_2 = c \]
• Try different values of \( c \)
• Evenly spaced
• Level sets together describe a hyperplane
• Hyperplane climbs in direction perpendicular to the linear level sets.
Example #2: Design of Radiation Therapy

<table>
<thead>
<tr>
<th>Area</th>
<th>Beam 1</th>
<th>Beam 2</th>
<th>Restrictions on Total Average Dosage (Kilorads)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Healthy anatomy</td>
<td>0.4</td>
<td>0.5</td>
<td>Minimize</td>
</tr>
<tr>
<td>Critical tissues</td>
<td>0.3</td>
<td>0.1</td>
<td>( \leq 2.7 )</td>
</tr>
<tr>
<td>Tumor region</td>
<td>0.5</td>
<td>0.5</td>
<td>= 6</td>
</tr>
<tr>
<td>Center of tumor</td>
<td>0.6</td>
<td>0.4</td>
<td>( \geq 6 )</td>
</tr>
</tbody>
</table>

Example #2: Design of Radiation Therapy

\[
\begin{align*}
\min_{x,y} & \quad P = 0.4x + 0.5y \\
\text{subject to:} & \quad 0.3x + 0.1y \leq 2.7 \\
& \quad 0.5x + 0.5y = 6 \\
& \quad 0.6x + 0.4y \geq 6 \\
& \quad x, y \geq 0
\end{align*}
\]

Example #2: Design of Radiation Therapy

\[
\begin{align*}
\min_{x,y} & \quad P = 0.4x + 0.5y \\
\text{subject to:} & \quad 0.3x + 0.1y \leq 2.7 \\
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Example #2: Design of Radiation Therapy
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\min_{x,y} P = 0.4x + 0.5y \\
\text{subject to:} \\
0.3x + 0.1y \leq 2.7 \\
0.5x + 0.5y = 6 \\
0.6x + 0.4y \geq 6 \\
x, y \geq 0
\]
Note: feasible region is a line segment!

Example #3: A Linear Program
\[
\max_{x,y} 20x + 9y \\
\text{subject to:} \\
3x + y \leq 20 \\
2x + y \leq 15 \\
x, y \geq 0
\]

Example #3: A Linear Program
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\max_{x,y} 20x + 9y \\
\text{subject to:} \\
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Example #3: A Linear Program

max \( 20x + 9y \)
subject to:
\[
\begin{align*}
3x + y &\leq 20 \\
2x + y &\leq 15 \\
x, y &\geq 0
\end{align*}
\]

(5,5)

Example #3: A Linear Program

max \( 20x + 9y \)
subject to:
\[
\begin{align*}
3x + y &\leq 20 \\
2x + y &\leq 15 \\
x, y &\geq 0
\end{align*}
\]

\[ y = \frac{20}{9}x + \frac{P}{9} \]

Example #3: A Linear Program

max \( 20x + 9y \)
subject to:
\[
\begin{align*}
3x + y &\leq 20 \\
2x + y &\leq 15 \\
x, y &\geq 0
\end{align*}
\]

(5,5)

Summary so far

- The linear inequalities form a simplex.
- The linear objective function creates a hyperplane.
- The solution is either:
  - a unique vertex or
  - a face of the simplex (more than one vertex).

Three Dimensions

The “simplex method” is a greedy algorithm that moves from one vertex of the feasible set (the simplex) to another until an optimum is attained.
Standard Form

- Converting an LP problem into “Standard Form” in two steps

Example

Minimize: \(-2x_1 + 3x_2 + x_3\)

Constraints:

\(-x_1 - x_2 - x_3 \geq -7\)
\(x_1 - 2x_2 + 3x_3 \leq 9\)
\(x_1, x_2 \geq 0\)
\(x_3 \leq 0\)

Conversion

As necessary:

- Multiply objective function by -1 to convert to Maximization
- Multiply constraint by -1 to convert to less-than-or-equal-to constraint
- Change of variable: Replace \(x_i\) by \(-x_i\) everywhere to create non-negativity constraint for \(x_i\)
- Equality constraint: replace with two inequalities: \(\leq \) & \(\geq\)

Matrix-Vector Notation

Maximize \(c^T x\)
subject to \(Ax \leq b\)
\(x \geq 0\)

\(c, b, A, x\) are coefficients in problems

Note: although our textbook defines standard form to involve the minimization of the objective, we are sticking with maximizing because that is what our version of simplex does (and frankly, that’s what the rest of ch. 7 of the textbook does as well).
Example: Matrix-Vector Notation

\[
\begin{align*}
\text{Maximize:} & \quad 2x_1 - 3x_2 + 4x_3, \\
\text{Constraints:} & \quad \begin{align*}
& x_1 + x_2 - x_3 \leq 7, \\
& x_1 - 2x_2 + 3x_3 \leq 5, \\
& x_1, x_2, x_3 \geq 0,
\end{align*}
\end{align*}
\]

\[
\begin{bmatrix}
 s_1 \\
 x_1 \\
 x_2 \\
 x_3
\end{bmatrix} = 
\begin{bmatrix}
 2 \\
 1 \\
 -3 \\
 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_1 \\
 a_2
\end{bmatrix} = 
\begin{bmatrix}
 1 & 1 & -1 \\
 1 & -2 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
 b \\
 c
\end{bmatrix} = 
\begin{bmatrix}
 7 \\
 4
\end{bmatrix}
\]

Example: Matrix-Vector Notation

\[
\begin{align*}
\text{Maximize:} & \quad 2x_1 - 3x_2 + 4x_3, \\
\text{Constraints:} & \quad \begin{align*}
& x_1 + x_2 - x_3 \leq 7, \\
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& x_1, x_2, x_3 \geq 0,
\end{align*}
\end{align*}
\]

Standard Form Step #2

- Step #2: Introduce slack variable \( s_i \) to produce an equality constraint from the \( i \)-th \( \leq \) constraint

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \rightarrow \quad s_i + \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad s_i \geq 0
\]

- Each resulting constraint must either be
  - An equality constraint (based on a \( \leq \) constraint from step #1)
  - Or: a non-negativity constraint

- Sometimes called “slack form”

Example: Standard Form Step #2

\[
\begin{align*}
\text{Maximize:} & \quad 2x_1 - 3x_2 + 4x_3, \\
\text{Constraints:} & \quad \begin{align*}
& x_1 + x_2 - x_3 \leq 7, \\
& x_1 - 2x_2 + 3x_3 \leq 5, \\
& x_1, x_2, x_3 \geq 0,
\end{align*}
\end{align*}
\]

- If your problem has
  - \( n \) variables
  - \( m \) constraints
    - (excluding the non-negativity constraints)
  - Introduce \( m \) slack variables

Example: Standard Form Step #2

\[
\begin{align*}
\text{Maximize:} & \quad 2x_1 - 3x_2 + 4x_3, \\
\text{Constraints:} & \quad \begin{align*}
& x_1 + x_2 - x_3 \leq 7, \\
& x_1 - 2x_2 + 3x_3 \leq 5, \\
& x_1, x_2, x_3 \geq 0,
\end{align*}
\end{align*}
\]

- Sometimes called “slack form”
Assignment

- HW #20 – due Friday
- Read Section 7.2 for Wednesday